

# Coplanarity In Twistor Space Of $\mathcal{N} = 4$ Next-To-MHV One-Loop Amplitude Coefficients

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Next-to-MHV one-loop amplitudes in  $\mathcal{N} = 4$  gauge theory can be written as a linear combination of known multivalued functions, called scalar box functions, with coefficients that are rational functions. We consider the localization of these coefficients in twistor space and prove that all of them are localized on a plane. The proof is done by studying the action of differential operators that test coplanarity on the unitarity cuts of the amplitudes.

## 1. Introduction

Perturbative amplitudes in  $\mathcal{N} = 4$  gauge theory possess many remarkable properties. One of them is that when transformed into twistor space [1], the amplitudes are localized on simple algebraic sets [2,3,4,5,6].

At tree level the algebraic sets can be thought of as unions of lines, or  $\mathbb{CP}^1$ 's, linearly embedded in  $\mathbb{CP}^3$  [4]. In [2], differential operators were introduced in order to study the support of the amplitudes directly in momentum space, without having to compute their twistor space transform. It turns out that at tree level, a straightforward application of these operators probes the structure of the amplitudes only for generic values of momenta, i.e., away from collinear or multi-particle singularities. For generic values of momenta, the twistor space picture simplifies; lines intersect in order to form connected quivers.

In the particular case of next-to-MHV amplitudes where three gluons have negative helicity and all other gluons have positive helicity, there are only two lines. Therefore, if the two lines intersect, we can say that the amplitude is localized on a plane.

At one-loop, the original proposal of [2] suggests that the localization on simple algebraic sets should hold. In [5], this structure was studied using the operators of [2], and the result did not seem to agree with the original picture. Motivated by the work of [7], this issue was reconsidered in [6], where an anomaly in the action of the operator was found to be responsible for the apparent disagreement. Once this anomaly is taken into account, the original picture is recovered [6,8,9,10].

One more important property of  $\mathcal{N} = 4$  amplitudes at one-loop is that they can be written as a sum over scalar box functions with rational functions as coefficients [11,12,13,14,15,16,17]. For next-to-MHV amplitudes, these coefficients can be efficiently calculated by using the holomorphic anomaly of unitarity cuts [9,18]. As an application of the method, the seven gluon amplitude with helicities  $(-, -, -, +, +, +, +)$  was computed. This result was independently obtained by the direct unitarity cut method in [19], along with the results for all other helicity configurations.

The method used in [9,18] to compute the coefficients has as a byproduct that the coefficients are localized on configurations in twistor space where some gluons lie on lines.

In [19], the twistor space localization of the coefficients was considered in detail. It was found that all coefficients of seven-gluon next-to-MHV amplitudes are localized on a plane. In addition, the coefficient of a certain class of three-mass box function was obtained to all multiplicities in next-to-MHV amplitudes with three adjacent minuses. For  $n \leq 10$ ,

these coefficients were found to be localized on a plane by numerical methods. Finally, the authors of [19] presented an outline of a proof that the coefficients of all next-to-MHV amplitudes are localized on a plane.

It is the aim of this paper to prove this statement. Our proof is significantly different from the argument of [19], but both are based on extending the arguments of [9], which shows that the coefficients are necessarily annihilated by some collinear operators, to include coplanar operators.

This paper is organized as follows: In section 2, we show that proving that the coefficients are annihilated by a certain differential operator is equivalent to proving that the operator produces a rational function when acting on certain unitarity cuts of the amplitude. This equivalence applies to all one-loop amplitudes. In section 3, we prove the latter statement for the coplanar operator acting on next-to-MHV amplitudes. Finally, in the appendix we prove that at tree-level, next-to-MHV amplitudes of gluons with at most two fermions or scalars are coplanar. This fact is used in the proof presented in section 3.

## 2. Preliminaries

Any leading-color  $n$ -gluon  $\mathcal{N} = 4$  amplitudes at one-loop can be written as a linear combination of scalar box functions as follows [15,17]:

$$A_{n;1}^{1\text{-loop}} = \sum_{i=1}^n \left( b_i F_{n;i}^{1m} + \sum_r c_{r,i} F_{n:r;i}^{2m\,e} + \sum_r d_{r,i} F_{n:r;i}^{2m\,h} + \sum_{r,r'} g_{r,r',i} F_{n:r:r';i}^{3m} + \sum_{r,r',r''} f_{r,r',r'',i} F_{n:r:r':r'';i}^{4m} \right). \quad (2.1)$$

The explicit form of these functions is not relevant in our discussion.<sup>1</sup> The coefficients are rational functions of the spinor inner products of external gluons. Recall that the momentum of each gluon can be written as  $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ . The inner products are defined as follows:  $\langle \lambda \lambda' \rangle = \epsilon_{ab} \lambda^a \lambda'^b$  and  $[\tilde{\lambda} \tilde{\lambda}'] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}^{\dot{a}} \tilde{\lambda}'^{\dot{b}}$ . We follow the conventions of [2].

We want to study the twistor space support of the coefficients. Doing so directly from the amplitude (2.1) is not simple. However, one of the observations in [9] is that from studying the action of collinear operators on the unitarity cuts of (2.1), one finds that the

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<sup>1</sup> See appendix A of [9] for a definition of these functions and a discussion of their discontinuities.

coefficients are annihilated by them. Here we want to generalize this argument starting with the following observation.

*Observation:* Let  $\mathcal{O}$  be any  $k$ -th order differential operator in the spinor variables. Let  $C_{i,i+1,\dots,j}$  denote the unitarity cut of (2.1) in the  $(i, i+1, \dots, j)$  channel. If  $\mathcal{O}C_{i,i+1,\dots,j}$  is a rational function, then  $\mathcal{O}(c) = 0$  for all coefficients  $c$  whose scalar box functions participate in this cut.

To prove this we make an argument similar to the one in [9] for collinear operators. Recall that the unitarity cut can be expressed two ways. One is by the cut integral, which in the  $(i, i+1, \dots, j-1, j)$ -channel is given by

$$C_{i,i+1,\dots,j-1,j} = \int d\mu A^{\text{tree}}((-\ell_1), i, i+1, \dots, j-1, j, (-\ell_2)) A^{\text{tree}}(\ell_2, j+1, j+2, \dots, i-2, i-1, \ell_1), \quad (2.2)$$

where  $d\mu$  is the Lorentz invariant phase space measure of two light-like vectors  $(\ell_1, \ell_2)$  constrained by momentum conservation.

The other way to write this unitarity cut is as the imaginary part of the amplitude in the regime where  $(p_i + p_{i+1} + \dots + p_j)^2 > 0$  and all other kinematical invariants are negative. This operation selects a subset of the scalar box functions, along with their proper coefficients. So, when the operator  $\mathcal{O}$  acts on a term of the cut, schematically we find that

$$\mathcal{O}C_{i,\dots,j} = \sum \mathcal{O}(c \text{Im} F). \quad (2.3)$$

Collect the terms where no derivatives act on  $\text{Im} F$ . This simply gives

$$\sum \mathcal{O}(c) \text{Im} F. \quad (2.4)$$

Now we use the fact that  $\text{Im} F$  is always the logarithm of a rational function (or for the four-mass box function, the logarithm of a function of the form  $A + \sqrt{B}$ , where  $A$  and  $B$  are rational functions). This implies that all terms where at least one derivative acts on the logarithm do not involve logarithms.

Now recall that  $\mathcal{O}C_{i,i+1,\dots,j}$  is a rational function by hypothesis. On the other hand, the terms in (2.4) have logarithms. As shown in [9], there is no way the logarithms can conspire to cancel among various box functions. Therefore,  $\mathcal{O}(c) = 0$  for each  $c$  in (2.4).

In the next section we specialize to the case of next-to-MHV amplitudes where  $\mathcal{O}$  is a second order differential operator that tests the coplanarity of four gluons in twistor space.

### 3. Proof Of Coplanarity Of Coefficients In NMHV Amplitudes

In this section, we prove that any coplanar operator in the external gluons annihilates all of the coefficients in (2.1) for a next-to-MHV amplitude. The idea is to show that any coplanar operator of four external gluons acting on a unitarity cut produces a rational function. It follows that the coefficients in that cut are coplanar, by the observation proven in section 2.

A coplanar operator is of the form [2]

$$K_{ijkl} = \langle i\ j \rangle [\tilde{\partial}_k\ \tilde{\partial}_l] + \langle j\ k \rangle [\tilde{\partial}_i\ \tilde{\partial}_l] + \langle k\ i \rangle [\tilde{\partial}_j\ \tilde{\partial}_l] + \langle k\ l \rangle [\tilde{\partial}_i\ \tilde{\partial}_j] + \langle i\ l \rangle [\tilde{\partial}_j\ \tilde{\partial}_k] + \langle j\ l \rangle [\tilde{\partial}_k\ \tilde{\partial}_i], \quad (3.1)$$

where

$$(\tilde{\partial}_i)_{\dot{\alpha}} = \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}}. \quad (3.2)$$

In the remainder of this section, we prove that  $K(C_{i,i+1,\dots,j-1,j})$  is a rational function using the integral form (2.2) of the unitarity cuts.

For an arbitrary next-to-MHV helicity assignment, the integral vanishes unless one of the two tree-level amplitudes in (2.2) is MHV [18]. The other tree amplitude will then always be next-to-MHV. It was shown explicitly in Appendix B of [18] that every coefficient in (2.1) can be calculated from some cut of this form, where the MHV side has at least three external gluons. The next-to-MHV side has at least four, otherwise it also becomes MHV or  $\overline{\text{MHV}}$ . (If one side of the cut has only three external gluons, it can be given an MHV assignment, or else it vanishes.) A special case arises when there are only six gluons, so that both of the tree amplitudes are MHV. This will not spoil our argument; simply exchange the two sides where appropriate.

Let us then decompose the cut (2.2) according to the helicity assignments of the cut propagators and treat each contribution separately. Each term takes the form

$$\int d\mu\ A_{\text{MHV}}^{\text{tree}}((- \ell_1), i, i+1, \dots, j-1, j, (- \ell_2)) A_{\text{NMHV}}^{\text{tree}}(\ell_2, j+1, j+2, \dots, i-2, i-1, \ell_1). \quad (3.3)$$

Our proof relies on the special properties of collinear and coplanar operators acting on tree-level MHV and next-to-MHV amplitudes. Recall that, for generic values of external momenta, collinear and coplanar operators annihilate MHV tree amplitudes of gluons [2], and coplanar operators annihilate next-to-MHV tree amplitudes also [4]. We review and extend the proof of these properties to the case where  $\ell_1$  and  $\ell_2$  are fermions or scalars in the appendix.

Now we are ready to show that a coplanar operator, acting on each term (3.3) of the cut, produces a rational function. Let  $m_1, m_2, \dots$  and  $n_1, n_2, \dots$  label external gluons on the MHV and NMHV sides of the cut, respectively. Indices  $a, b, c, d$  will label any gluons on either side. Since the coplanar operator is antisymmetric under index exchange, there are five cases we must consider:  $K_{m_1 m_2 m_3 m_4}, K_{m_1 m_2 m_3 n}, K_{m_1 m_2 n_1 n_2}, K_{m n_1 n_2 n_3}, K_{n_1 n_2 n_3 n_4}$ .

For the first two cases,  $K_{m_1 m_2 m_3 m_4}$  and  $K_{m_1 m_2 m_3 n}$ , there is a useful identity that expresses the simple geometrical fact that any three points that are collinear are coplanar with any fourth point. This identity can be written in the following form:

$$K_{abcd} = [F_{abc} \tilde{\partial}_d] + \frac{1}{\langle a c \rangle} \left( \langle d a \rangle [F_{abc} \tilde{\partial}_c] + \langle c d \rangle [F_{abc} \tilde{\partial}_a] \right). \quad (3.4)$$

We see that the collinear operator  $F_{m_1 m_2 m_3}$  will appear in every term of (3.4) for the operators  $K_{m_1 m_2 m_3 m_4}$  and  $K_{m_1 m_2 m_3 n}$ . But the action of this collinear operator is familiar [6]: it acts only on the MHV side of the cut, where it annihilates  $A_{\text{MHV}}^{\text{tree}}$  except for the holomorphic anomaly, which produces a delta function localizing the integral to give a rational function.

For the fifth case,  $K_{n_1 n_2 n_3 n_4}$ , the derivatives  $\tilde{\partial}_n$  act trivially on the measure  $d\mu$ , so this coplanar operator passes through the measure and the MHV amplitude. To see this, note that the measure can be written as follows:

$$d\mu = \delta^{(+)}(\ell_1^2) \delta^{(+)}(\ell_2^2) \delta^{(4)}(\ell_1 + \ell_2 - P_L), \quad (3.5)$$

where  $P_L = p_i + p_{i+1} + \dots + p_j$ . If we use conformal invariance to set the coordinates  $Z = (\lambda_1, \lambda_2, \mu_1, \mu_2)$  in twistor space of  $n_1$  and  $n_2$  to  $Z_{n_1} = (1, 0, 0, 0)$  and  $Z_{n_2} = (0, 1, 0, 0)$ , then, as described in rigorous detail in section 3.3 of [2],

$$K_{n_1 n_2 n_3 n_4} = [\tilde{\partial}_{n_3} \tilde{\partial}_{n_4}]. \quad (3.6)$$

Moreover,  $\lambda$  and  $\tilde{\lambda}$  of the gluons in  $P_L$  and  $n_3$  and  $n_4$  are all independent variables, since momentum conservation is satisfied by making  $\tilde{\lambda}_{n_1}$  and  $\tilde{\lambda}_{n_2}$  depend on all other gluons.

Now we can prove that  $K_{n_1 n_2 n_3 n_4}$  produces a rational function when acting on the cut integral  $C_{i, i+1, \dots, j}$  given in (2.2). The proof is as follows. We have shown that  $K_{n_1 n_2 n_3 n_4}$  when acts on  $C_{i, i+1, \dots, j}$  only affects the NMHV amplitude  $A^{\text{tree}}(\ell_2, j+1, \dots, i-1, \ell_1)$ . The particles running in the cut propagators, i.e.,  $\ell_1$  and  $\ell_2$ , can be gluons, fermions or scalars. In any of these cases we prove in the appendix that for generic values of  $p_{\ell_2}, p_{j+1}, \dots, p_{i-1}, p_{\ell_1}$ , the operator  $K_{n_1 n_2 n_3 n_4}$  annihilates the amplitude. Since we are

assuming that the values of  $p_{j+1}, \dots, p_{i-2}$ , and  $p_{i-1}$  are generic and that the cut is in at least a three-particle channel, the only possibility of a nonzero contribution is at isolated points in the phase space integral over  $\ell_1$  and  $\ell_2$  [6]. This implies that the integral localizes and can produce at most a rational function, since the tree-level amplitudes in the integrand are rational.

Finally, there are the third and fourth cases,  $K_{m_1 m_2 n_1 n_2}$  and  $K_{m n_1 n_2 n_3}$ . We will apply the results from the other cases to prove that these operators, too, produce rational functions.

The terms of a coplanar operator (3.1) are of three types:  $f_1[\tilde{\partial}_{m_1} \tilde{\partial}_{m_2}]$ ,  $f_2[\tilde{\partial}_m \tilde{\partial}_n]$ , and  $f_3[\tilde{\partial}_{n_1} \tilde{\partial}_{n_2}]$ . The coefficients  $f_r$  of the differential operators are already rational, so we consider only the action of the operators  $\tilde{\partial}$ . We will be able to prove that in fact *each* of these terms acts on the cut to produce a rational function.

Now, for terms of the form  $f_1[\tilde{\partial}_{m_1} \tilde{\partial}_{m_2}]$  (the first type of term mentioned above), use conformal invariance to set  $Z_{m_3} = (1, 0, 0, 0)$  and  $Z_n = (0, 1, 0, 0)$ , where  $m_3$  is different from  $m_1$  and  $m_2$ . This is possible because the MHV side has at least three external gluons. Then we find that

$$[\tilde{\partial}_{m_1} \tilde{\partial}_{m_2}] = K_{m_1 m_2 m_3 n}. \quad (3.7)$$

This is an operator we have already considered.

For terms of the form  $f_2[\tilde{\partial}_{m_1} \tilde{\partial}_n]$ , we can make a similar argument, using two additional gluons  $m_2, m_3$  with  $Z_{m_2} = (1, 0, 0, 0)$  and  $Z_{m_3} = (0, 1, 0, 0)$ , so that

$$[\tilde{\partial}_{m_1} \tilde{\partial}_n] = K_{m_1 n m_2 m_3}, \quad (3.8)$$

which again is a coplanar operator we have already considered.

Finally, there are terms of the form  $f_3[\tilde{\partial}_{n_1} \tilde{\partial}_{n_2}]$ . Since the next-to-MHV amplitude must have at least four external gluons, we may choose two others  $n_3, n_4$  and set  $Z_{n_3} = (1, 0, 0, 0)$  and  $Z_{n_4} = (0, 1, 0, 0)$ , so that

$$[\tilde{\partial}_{n_1} \tilde{\partial}_{n_2}] = K_{n_1 n_2 n_3 n_4}, \quad (3.9)$$

which again is a coplanar operator we have already considered.

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## Appendix A. Coplanarity Of Next-to-MHV Tree-Level Amplitudes

In this appendix we review the proof that all next-to-MHV tree-level amplitudes of gluons are localized on a plane. We extend the proof to include  $\mathcal{N} = 4$  amplitudes with at most two external fermions or two external scalars.

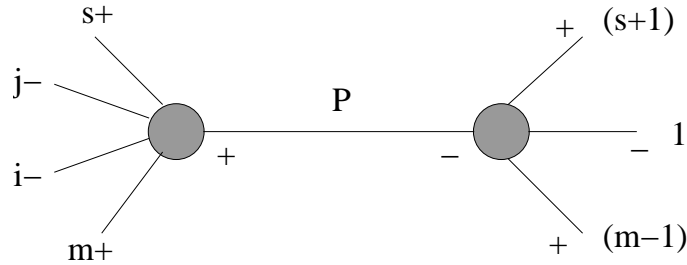
All these amplitudes can be computed using extensions of the MHV diagrams of [4] to the case of fermions and scalars [20,21,22,23]. The basic idea is to use MHV vertices continued off-shell and connected by propagators. Each MHV vertex contains at most two fermions or two scalars and can easily be computed in terms of an MHV vertex of only external gluons by using one of the following Ward identities:

$$\begin{aligned} A(F_1^-, g_2^+, \dots, g_i^-, \dots, F_n^+) &= \frac{\langle j \ n \rangle}{\langle j \ 1 \rangle} A^{\text{MHV}}(g_1^-, g_2^+, \dots, g_j^-, \dots, g_n^+), \\ A(S_1^-, g_2^+, \dots, g_i^-, \dots, S_n^+) &= \frac{\langle j \ n \rangle^2}{\langle j \ 1 \rangle^2} A^{\text{MHV}}(g_1^-, g_2^+, \dots, g_j^-, \dots, g_n^+). \end{aligned} \quad (\text{A.1})$$

From this we conclude that all NMHV amplitudes can be written as a sum of MHV diagrams of gluons with prefactors that depend only on the holomorphic spinors. Therefore, proving that  $K$  annihilates the amplitude is equivalent to proving that  $K$  annihilates any of the MHV diagrams that involve only gluons and two MHV vertices.

In general, the twistor space localization of tree amplitudes of gluons was considered in section 2 of [5]. Here we review the arguments of [5] for the particular case of tree-level next-to-MHV amplitudes.

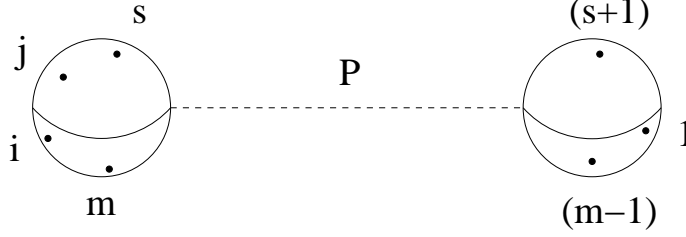
A general NMHV amplitude of the form  $A_n^{\text{tree}}(1^-, 2^+, \dots, i^-, \dots, j^-, \dots, n)$  can be computed by adding up all possible MHV diagrams with one link and two nodes [4].



**Fig. 1:** An MHV diagram contributing to the NMHV amplitude  $A_n^{\text{tree}}$ .

In [4], these diagrams were shown to be computed from a twistor space calculation where gluons are separated into two groups, with one negative-helicity gluon in one group and two in the other. An example is shown in Figure 1. Now, each group is localized on a line and the two lines are connected by a propagator, as shown in Figure 2. The two lines do not have to intersect.





**Fig. 2:** Twistor diagram that corresponds to the MHV diagram shown in fig. 2.

From the twistor construction it is clear that all gluons in each group are collinear.

As mentioned above the two lines do not necessarily intersect. The reason is that the Feynman propagator we use is not well defined unless an  $i\epsilon$  prescription is chosen. The conclusion we want to get does not depend on the choice so let us just take as our definition  $1/(P^2 + i\epsilon)$ , where  $P = p_m + p_{m+1} + \dots + p_s$ ; see Figure 2. This can be written as the principal value plus a delta function with support at  $P^2 = 0$ . Now, the Fourier transform of the principal part of  $1/(P^2 + i\epsilon)$  into coordinate space is a delta function localized at points where  $(x - y)^2 = 0$ , where  $x$  is the spacetime position of one MHV vertex and  $y$  the position of the other. Therefore, the contribution from the principal value gives diagrams at points in Minkowski space that are on a light ray. It turns out that a light ray in twistor space corresponds to a point [1]. Recalling that the MHV vertices correspond to lines in twistor space, this implies that the two lines intersect. Therefore all gluons are coplanar.

Now we have to worry about the delta function localized at  $P^2 = 0$ . In general we do not discuss these terms because we consider external gluons with generic momenta and  $P^2 \neq 0$ . However, in our case,  $P$  might contain  $\ell_1$  or  $\ell_2$ , which are integration variables. It turns out that even in this case  $P^2 \neq 0$ . To see this, consider  $s = \ell_1$  and  $s + 1 = \ell_2$  in the example of Figures 1 and 2. Then  $P = p_m + p_{m+1} + \dots + p_{s-1} + \ell_1$ . The measure in the cut integral has delta functions with support at

$$\begin{aligned} \ell_1^2 = \ell_2^2 = 0, \\ \ell_1 + \ell_2 = -(p_{s+2} + \dots + p_{m-1} + p_m + \dots + p_{s-1}). \end{aligned} \tag{A.2}$$

The latter is essentially the momentum conservation constraint for the NMHV amplitude.

Using (A.2) to solve for  $\ell_2$  and imposing that  $\ell_2^2 = 0$ , we find that  $R^2 = -2\ell_1 \cdot R$ , where  $R = (p_{s+2} + \dots + p_{m-1} + p_m + \dots + p_{s-1})$ . On the other hand, from  $P^2 = 0$  we get  $Q^2 = -2\ell_1 \cdot Q$ , where  $Q = p_m + \dots + p_{s-1}$ .

Because we take generic values of the external momenta  $p_{s+2}, \dots, p_{m-1}$ , the variables  $Q$  and  $R$  are independent, and the two equations will either localize the integral, producing a rational function, or else cannot be satisfied simultaneously.

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